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(6) POLYEXTREMAL PRINCIPLES AND SEPARABLY-INFINITE PROGRAMS

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ABSTRACT

As a direct extension of Charnes' characterization of two-person zero-sum constrained games by linear programming, ~~we~~ it is shown how a general class of saddle value problems can be reduced to a pair of uniextremal dual separably-infinite programs. These programs have an infinite number of variables and an infinite number of constraints, but only a finite number of variables appear in an infinite number of constraints and only a finite number of constraints have an infinite number of variables. The conditions under which the characterization holds are among the more general ones appearing in the literature sufficient to guarantee the existence of a saddle point of a concave-convex function.

The key construction involves augmenting a given player's original set of variables by generalized finite sequences determined by the other player's constraint set and objective function. A duality theory is developed which includes complementarity conditions, thereby making contact with the numerical treatment of semi-infinite programming.

Key Words. Polyextremal Problems, Saddle Values, Separably-Infinite Programming, Generalized Finite Sequence Spaces, Moment Cones, Duality and Complementarity.

POLYEXTREMAL PRINCIPLES AND SEPARABLY INFINITE PROGRAMS

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I. Introduction: Constrained Saddle Value Problems

In 1953, the first author in [3] showed how directly from the data a bilinear saddle value problem with polyhedral constraints could be replaced with a pair of uniextremal dual linear programming problems. The basic saddle problem is the following one. *For a given* $m \times n$ *matrix* A,

find
$$\max_{p \in \mathbb{R}^m} \min_{q \in \mathbb{R}^n} p^T A q$$

subject to

(I)	(II)
$\sum_{i=1}^{m} p_i = 1$	$\sum_{j=1}^{n} q_j = 1$
$p^T D \leq d^T$	$Bq \geq b$
$p \geq 0$	$q \geq 0,$

where D *is an* $m \times r$ *matrix*, B *is an* $s \times n$ *matrix*, $d \in \mathbb{R}^r$, *and* $b \in \mathbb{R}^s$.

The equivalent pair of dual linear programs is:

$$
\begin{array}{ll}
\text{I} & \text{II} \\
\max \alpha + y^T b & \min \delta + d^T x \\
\alpha \in \mathbb{R},\ p \in \mathbb{R}^m,\ y \in \mathbb{R}^s & \delta \in \mathbb{R},\ q \in \mathbb{R}^n,\ x \in \mathbb{R}^r \\
\alpha e^T_{(n)} - p^T A + y^T B \leq 0 & \delta e_{(m)} - Aq + Dx \geq 0 \\
\qquad p^T e_{(m)} = 1 & \qquad e^T_{(n)} q = 1 \\
\qquad p^T D \leq d^T & \qquad Bq \geq b \\
\qquad p, y \geq 0 & \qquad q, x \geq 0,
\end{array}
$$

where $e_{(n)}$ consists of n ones, $e_{(m)}$ consists of m ones.

Charnes posed the problem of finding a similar reduction of biextremality to uniextremality, since for more general situations in many fields of physics, engineering, economics, etc., one can easily obtain biextremal characterizations, although no uniextremal principle is apparent.

While existing work of Danskin [7], Gol'stein [12], Rockafellar [15], Stoer-Witzgall [16] and others has shown how to reduce the study of concave-convex saddle functions to pairs of dual convex programming problems, except in the simplest of cases when the internal extremizations can be explicitly carried out, no reduction in the number of extremizations has been accomplished. And, as the work of Bracken, Falk, and McGill [1] and Bracken and McGill [2] shows, explicit analysis and computation of such convex programming problems with embedded extremizations is neither transparent nor facile. Thus, while such constructions may be useful in establishing important general properties of the saddle value problem, with them, as Poincaré might say, the problem posed is "very little solved".

By means of a new construct, "separably-infinite programming" [6], we have solved the problem for a general kind of separability and a "functional" bilinearity in the saddle function* which is equivalent to:

$$\underline{\text{find}} \quad \sup_{p\in P} \inf_{q\in Q}\{g(p) + p^T A q + h(q)\}$$

<u>where</u> P <u>and</u> Q <u>are arbitrary closed convex sets in</u> $\mathbb{R}^m$ <u>and</u> $\mathbb{R}^n$, <u>respectively</u>, <u>and where</u> g <u>is concave and</u> h <u>is convex</u>.

We show that this nonlinear polyextremal problem is equivalent to a pair of uniextremal dual separably-infinite programs under assumptions which are among the most general ones appearing in the literature for saddle value problems.

Our solution to this nonlinear saddle problem actually rests upon finite linear programming methods in the sense that when appropriate finite discretizations are made of the problem, than an approximate solution is obtained by solving the classical constrained game case. The bridge between the finite linear programming problems so obtained and the infinite structure of the original nonlinear saddle problem is a class of generalized

---

*By this we mean saddle functions of the form

$$\sum_{j=1}^{n'} \sum_{i=1}^{m'} g_i(p) a_{ij} h_j(q),$$

where g_i is closed concave on $\mathbb{R}^m$, $i = 1,\ldots,m'$, h_j is closed convex on $\mathbb{R}^n$, $j = 1,\ldots,n'$, and $a_{ij} \geq 0$ for all i and j.

finite sequences, as employed in the theory of semi-infinite programming. Generalized finite sequences permit the finite discretizations themselves to vary freely, as a class of probability measures with finite support subject to variation. In this sense we obtain linearizations of the nonlinear poly-extremal problem.

As in the classical two-person zero-sum game and its constrained game extension, our approach does not require the specification of any perturbations which for example, in a functional transform approach would necessarily be required in advance of duality developments. As in elementary finite linear programming, one need not be concerned about perturbations for duality purposes. They are handled automatically.

In this context we show that the decision vector set of one player is augmented by generalized finite sequences determined by (a), the additional convex constraints on the other player's variables and (b), the objective function or its "epigraph" defined over the other player's original decision vectors. Thus, player I's variables shall consist of the p-vectors plus generalized finite sequences determined by linear inequality representations of player II's constraint set Q and player II's objective function h. The situation is a direct extension of the duality obtained in the finite and classical constrained game case. We illustrate the construction with a simple numerical example in an economic context.

The particular approach to the study of saddle value problems provides the opportunity for numerical treatment by semi-infinite programming methods [11], [13]. In particular, the authors in [6] have developed a system of nonlinear equations obtained from separably infinite duality relationships to which Newton type methods apply.

We now formally introduce the polyextremal problems and set forth the main assumptions upon which our approach is based.

2. Assumptions and Definitions Underlying the Polyextremal Problems

The problem data are the following.

(i) a closed concave function g in m variables, $p \in \mathbb{R}^m$,

(ii) a closed convex function h in n variables, $q \in \mathbb{R}^n$,

(iii) an $m \times n$ matrix A,

(iv) an explicit closed convex constraint set for the p-variables given in linear inequality form,

$$P = \{p \in \mathbb{R}^m \mid p^T d(\alpha) \leq d_{m+1}(\alpha),\ d(\alpha) \in \mathbb{R}^m,\ d_{m+1}(\alpha) \in \mathbb{R},\ \text{for all } \alpha \text{ in a set } \mathcal{R}\},$$

and

(v) an explicit closed convex constraint set for the q-variables given in linear inequality form,

$$Q = \{q \in \mathbb{R}^n \mid b(\beta)^T q \geq b_{n+1}(\beta),\ b(\beta) \in \mathbb{R}^n,\ b_{n+1}(\beta) \in \mathbb{R}\ \text{for all } \beta \text{ in a set } \mathcal{S}\}.$$

In this paper we investigate the following two polyextremal problems:

<u>find</u> $$V_M = \sup_{p \in P} \inf_{q \in Q} \{g(p) + p^T A q + h(q)\}$$

and

<u>find</u> $$V_N = \inf_{q \in Q} \sup_{p \in P} \{g(p) + p^T A q + h(q)\}.$$

As a closed convex set in $\mathbb{R} \times \mathbb{R}^m$, the <u>hypograph of</u> g, $\{(p_o, p) \in \mathbb{R} \times \mathbb{R}^m \mid p_o \leq g(p)\}$, can be characterized by a system of supporting hyperplanes of the form:

$$\{(p_o,p) \in \mathbb{R} \times \mathbb{R}^m | v_o(\sigma)p_o + v(\sigma)^T p \le v_{m+1}(\sigma),\ v(\sigma)\in\mathbb{R}^m,$$
$$v_o(\sigma),\ v_{m+1}(\sigma)\in\mathbb{R},\ v_o(\sigma) \ge 0 \quad \text{for all} \quad \sigma \quad \text{in a}$$
$$\text{set} \quad \mathcal{J}\}. \tag{1}$$

The <u>epigraph of the closed convex function</u> h, $\{(q_o,q) \in \mathbb{R} \times \mathbb{R}^n | h(q) \le q_o\}$ can be characterized by a system of the form:

$$\{(q_o,q) \in \mathbb{R} \times \mathbb{R}^n | u_o(\gamma)q_o + u(\gamma)^T q \ge u_{n+1}(\gamma),\ u(\gamma)\in\mathbb{R}^n,$$
$$u_o(\gamma),\ u_{n+1}(\gamma)\in\mathbb{R},\ u_o(\gamma) \ge 0 \quad \text{for all} \quad \gamma \quad \text{in a set} \quad \mathcal{U}\}. \tag{2}$$

<u>Definition</u>. A supporting hyperplane of (1)

$$v_o(\sigma)p_o + v(\sigma)^T p = v_{m+1}(\sigma)$$

is termed a <u>domain constraining vertical hyperplane</u> if and only if $v_o(\sigma) = 0$, $v(\sigma) \ne \underset{\sim}{0}$ and a positive multiple of $v(\sigma)$, $v_{m+1}(\sigma)$ equals $d(\alpha)$, $d_{m+1}(\alpha)$ for some $\alpha \in \mathcal{R}$. [In general $\underset{\sim}{0}$ shall indicate a zero vector of appropriate, compatible dimension.]

Similarly a domain constraining vertical hyperplane in (2) is one which is a positive multiple of the vector $b(\beta)$, $b_{n+1}(\beta)$ for some $\beta \in \mathcal{S}$.

The following assumptions will prevail throughout the paper.

<u>Assumption 1</u>. Domain $g \cap P \ne \emptyset$, domain $h \cap Q \ne \emptyset$, and the following subsets of P and Q respectively are non-empty,

$$P_{\infty} = \{P \cap \operatorname{dom} g \mid \inf_{q \in Q}\{p^T A q + h(q)\} > -\infty\}$$

$$Q_{\infty} = \{Q \cap \operatorname{dom} h \mid \sup_{p \in P}\{g(p) + p^T A q\} < +\infty\}.$$

Assumption 2. Any vertical hyperplane in either of the supporting hyperplane systems (1) or (2) is domain constraining.

Assumption 3.1. The convex cone in $\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}$ generated by

$$\{v_o(\sigma), v(\sigma), -v_{m+1}(\sigma)\}_{\sigma \in \mathcal{J}} \cup \{0, d(\alpha), -d_{m+1}(\alpha)\}_{\alpha \in \mathcal{R}} \cup \{0, \underset{\sim}{0}, -1\}$$

is closed.

Assumption 3.2. The convex cone in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ generated by

$$\{u_o(\gamma), u(\gamma), -u_{n+1}(\gamma)\}_{\gamma \in \mathcal{U}} \cup \{0, b(\beta), -b_{n+1}(\beta)\}_{\beta \in \mathcal{S}} \cup \{0, \underset{\sim}{0}, 1\}$$

is closed.

We turn now to the main construction of this paper, namely obtaining two dual infinite linear programs stemming from V_M and V_N.

3. Constructing Separably Infinite Programs from the Polyextremal Problems

Program I. Find $V_I =$

$$\sup \sum_{\beta} y(\beta) b_{n+1}(\beta) + \sum_{\gamma} u_{n+1}(\gamma)\lambda(\gamma) + p_o \tag{3}$$

from among $(p_o, p) \in \mathbb{R} \times \mathbb{R}^m$, $y \in \mathbb{R}^{(\mathcal{B})}$, $\lambda \in \mathbb{R}^{(\mathcal{U})}$, generalized finite sequence spaces, subject to

$$p^T d(\alpha) \leq d_{m+1}(\alpha), \text{ all } \alpha \in \mathcal{R} \tag{4}$$

$$p_o v_o(\sigma) + p^T v(\sigma) \leq v_{m+1}(\sigma), \text{ all } \sigma \in \mathcal{J} \tag{5}$$

$$\sum_{\beta} y(\beta) b(\beta)^T + \sum_{\gamma} \lambda(\gamma) u(\gamma)^T - p^T A = \underset{\sim}{0}^T \tag{6}$$

$$\sum_{\gamma} \lambda(\gamma) u_o(\gamma) = 1 \tag{7}$$

and

$$y \geq 0, \quad \lambda \geq 0. \tag{8}$$

In general, a generalized finite sequence space with respect to a set W, denoted $\mathbb{R}^{(W)}$, is the linear space of all real valued functions on W having only finitely many non-zero images.

Program I is a special case of Program P of the Appendix, whose dual D becomes Program II (in the notation of Program I).

<u>Program</u> <u>II</u>. <u>Find</u> $V_{II} =$

$$\underline{\inf} \quad q_o + \sum_\alpha d_{m+1}(\alpha)x(\alpha) + \sum_\sigma v_{m+1}(\sigma)\eta(\sigma) \tag{9}$$

<u>from</u> <u>among</u> $(q_o,q) \in \mathbb{R} \times \mathbb{R}^n,\ x\in\mathbb{R}^{(\mathcal{R})},\ \eta\in\mathbb{R}^{(\mathcal{J})}$

<u>subject</u> <u>to</u>

$$b(\beta)^T q \geq b_{n+1}(\beta),\ \underline{\text{all}}\quad \beta\in\mathcal{B} \tag{10}$$

$$u_o(\gamma)q_o + u(\gamma)^T q \geq u_{n+1}(\gamma),\ \underline{\text{all}}\quad \gamma\in\mathcal{U} \tag{11}$$

$$\sum_\sigma v_o(\sigma)\eta(\sigma) = 1 \tag{12}$$

$$-Aq + \sum_\alpha d(\alpha)x(\alpha) + \sum_\sigma v(\sigma)\eta(\sigma) = \underset{\sim}{0} \tag{13}$$

<u>and</u>

$$x \geq 0, \quad \eta \geq 0. \tag{14}$$

According to the duality developments reviewed in the Appendix, see also [6], Programs I and II satisfy the duality inequality, namely if $\{(p_o,p),y,\lambda\}$ is I-feasible and $\{(q_o,q),x,\eta\}$ is II-feasible, then

$$\sum_\beta y(\beta)b_{n+1}(\beta) + \sum_\gamma u_{n+1}(\gamma)\lambda(\gamma) + p_o$$
$$\leq q_o + \sum_\alpha d_{m+1}(\alpha)x(\alpha) + \sum_\sigma v_{m+1}(\sigma)\eta(\sigma). \tag{15}$$

The next task is to develop relationships between $p\in P_\infty$ introduced in Assumption 1 and feasible lists $\{(p_o,p),y,\lambda\}$ of Program I, and similarly for Program II. Actually, these relationships are needed in order to establish inequalities among all four program values, V_M, V_N, V_I, and V_{II}.

Lemma 1. Assume that P_∞ is not empty and that Assumption 3.2 holds. Then

(i) any $p \in P_\infty$ is extendable to a feasible list $\{(p_o,p),y,\lambda\}$ of Program I, and

(ii) if $\{(p_o,p),y,\lambda\}$ is any I-feasible list, then $p \in P_\infty$.

In particular, Program I is consistent.

Proof. (i) Let $p \in P_\infty$ and assume to the contrary that it is not extendable. This assumption implies that the following program is inconsistent:

Program A

$$V_A = \sup \sum_\beta y(\beta) b_{n+1}(\beta) + \sum_\gamma u_{n+1}(\gamma)\lambda(\gamma)$$

subject to
$$\sum_\beta y(\beta) b(\beta) + \sum_\gamma u(\gamma)\lambda(\gamma) = p^T A$$

$$\sum_\gamma u_o(\gamma)\lambda(\gamma) = 1$$

and $y \geq 0$, $\lambda \geq 0$.

The dual to the above program is:

Program B

$$V_B = \inf q_o + p^T A q$$

subject to
$$b(\beta)^T q \geq b_{n+1}(\beta), \quad \text{all } \beta \in \mathcal{S} \tag{16a}$$

$$u_o(\gamma) q_o + u(\gamma)^T q \geq u_{n+1}(\gamma), \quad \text{all } \gamma \in \mathcal{U}. \tag{16b}$$

The inequalities (16a), (16b) are consistent because domain $h \cap Q \neq \emptyset$ and the fact that (16b) is a supporting hyperplane representation of the epigraph of h. Furthermore, for any feasible (q_o,q) we have

$$q_o + p^T Aq \geq h(q) + p^T Aq \geq \inf_{q \in Q}\{p^T Aq + h(q)\} > -\infty$$

since $p \in P_\infty$. Therefore, V_B is finite. It therefore follows from the closure of the moment cone determined by (16a) and (16b) (Assumption 3.2) that Program A is consistent, which is a contradiction. Therefore any $p \in P_\infty$ is extendable to a feasible list for Program I.

(ii) Assume that $\{(\bar{p}_o,\bar{p}),\bar{y},\bar{\lambda}\}$ is I-feasible and assume to the contrary that $\bar{p} \notin P_\infty$. Nevertheless, (16a) and (16b) are consistent for the same reasons as above in part (i), but now $V_B = -\infty$ (since $\bar{p} \notin P_\infty$ by assumption). In this case the closure of the relevant moment cone (Assumption 3.2) implies that Program A is inconsistent. However, setting $y = \bar{y}$, $\lambda = \bar{\lambda}$ yields a feasible list for A, which is a contradiction. It therefore follows that $p \in P_\infty$.

The completely symmetric and analogous result holds for Program II under Assumption 3.1 with respect to the set Q_∞. We shall therefore view this case as also part of the statement of Lemma 1. The next lemma presents what we shall term "poly-extremal duality inequalities."

<u>Lemma</u> <u>2</u>. Let Assumptions 1, 2, and 3 prevail. Then

$$V_I \le \sup_P \inf_Q \{g(p) + p^T A q + h(q)\} \le V_{II}$$

and

$$V_I \le \inf_Q \sup_P \{g(p) + p^T A q + h(q)\} \le V_{II}.$$

In particular, V_I and V_{II} are finite valued.

<u>Proof</u>. Since $P_\infty \neq \emptyset$, I is consistent by Lemma 1, and so let $\{(p_o,p),y,\lambda\}$ be any I-feasible list. It suffices to take $p_o = g(p)$.

Given any $q \in Q \cap \text{dom } h$, set $q_o = h(q)$. Multiplying (6) through by q and using (10), (11) and the non-negativity of y,λ yields:

$$\begin{aligned} p^T A q &= \sum_\beta y(\beta) b(\beta)^T q + \sum_\gamma \lambda(\gamma) u(\gamma)^T q \\ &\ge \sum_\beta y(\beta) b_{n+1}(\beta) + \sum_\gamma u_{n+1}(\gamma)\lambda(\gamma) - \sum_\gamma u_o(\gamma) q_o \lambda(\gamma). \end{aligned}$$

But the term to the right of the inequality is

$$\sum_\beta y(\beta) b_{n+1}(\beta) + \sum_\gamma u_{n+1}(\gamma)\lambda(\gamma) - h(q),$$

using (7). Therefore,

$$g(p) + \sum_\beta y(\beta) b_{n+1}(\beta) + \sum_\gamma u_{n+1}(\gamma)\lambda(\gamma) \le g(p) + p^T A q + h(q). \quad (17)$$

Since the q employed in (17) was arbitrarily chosen in $Q \cap \text{dom } h$, it follows that

$$g(p) + \sum_{\beta} y(\beta) b_{n+1}(\beta) + \sum_{\gamma} u_{n+1}(\gamma)\lambda(\gamma) \le \inf_{q\in Q}\{g(p) + p^T Aq + h(q)\}. \quad (18)$$

Therefore, (18) holds for every $\{(p_o,p),y,\lambda\}$ feasible for I and hence by Lemma 1, for every $p\in P_\infty$. Therefore,

$$V_I \le \sup_{p\in P_\infty} \inf_{q\in Q}\{g(p) + p^T Aq + h(q)\} = \sup_{p\in P} \inf_{q\in Q}\{g(p) + p^T Aq + h(q)\}, \quad (19)$$

the latter equality stemming from non-emptiness of P_∞.

On the other hand (17) yields an inequality on the respective suprema, namely,

$$V_I \le \sup_{p\in P}\{g(p) + p^T Aq + h(q)\}$$

for each $q\in Q$. Hence

$$V_I \le \inf_{q\in Q} \sup_{p\in P}\{g(p) + p^T Aq + h(q)\}. \quad (20)$$

A completely analogous development involving Program II and $Q_\infty \ne \emptyset$ yields:

(a), the analog of (19) namely

$$V_{II} \ge \inf_{q\in Q} \sup_{p\in P}\{h(q) + p^T Aq + g(p)\}, \quad (21)$$

and

(b), the analog of (20) namely

$$V_{II} \ge \sup_{p\in P} \inf_{q\in Q}\{h(q) + p^T Aq + g(p)\}. \quad (22)$$

The set of inequalities (18), (19), (21), and (22) give the required statements of the lemma.

4. The Main Duality Theorems

We are now ready to translate the duality results given in the Appendix to Programs I and II. First, we introduce the two convex cones stemming respectively from the linear inequality representations of the convex sets P and Q.

C_P is the convex cone spanned by

$$\begin{pmatrix} d(\alpha) \\ -d_{m+1}(\alpha) \end{pmatrix}_{\alpha\in\mathcal{R}} \cup \begin{pmatrix} 0 \\ -1 \end{pmatrix} , \tag{23}$$

while C_Q is the convex cone spanned by

$$\begin{pmatrix} b(\beta) \\ -b_{n+1}(\beta) \end{pmatrix}_{\beta\in\mathcal{S}} \cup \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{24}$$

Theorem 1. Let Assumptions 1-3 prevail. Assume that $(p_o,p) \in \mathbb{R} \times \mathbb{R}^m$, $p \in 0^+P$, $(g0^+)(p) \geq p_o$, and $\begin{pmatrix} A^T p \\ p_o \end{pmatrix} \in C_Q$ implies $(p_o,p) = \underset{\sim}{0}$.

Then $V_I = V_M = V_N = V_{II}$ and V_I is a maximum.

Proof. By Lemmas 1 and 2 it follows that both Programs I and II are consistent and finite valued. The theorem is a specialization of Theorem A1 in the Appendix and so the main task is to define the sets (a) 0^+K_Q and (b) $\{y\in\mathbb{R}^m | \begin{pmatrix} A^T y \\ -b^T y \end{pmatrix} \in C_S\}$ used in (A1) there for Program I. The fact that the only vertical hyperplanes permitted in support systems of the functions g and h are domain constraining will also be used.

The recession cone in (a) becomes the set of those $(p_o,p) \in \mathbb{R} \times \mathbb{R}^m$ which satisfy

$$p^T d(\alpha) \leq 0, \quad \text{all} \quad \alpha \in \mathcal{R} \tag{25}$$

and

$$p_o v_o(\sigma) + p^T v(\sigma) \leq 0, \quad \text{all} \quad \sigma \in \mathfrak{J}. \tag{26}$$

Now (25) is equivalent to $p \in 0^+P$ while (26) is merely $(p_o, p) \in 0^+(\text{hypo } g)$. Since $0^+(\text{hypo } g) = \text{hypo}(g0^+)$, (26) is equivalent to

$$(g0^+)(p) \geq p_o.$$

Translating the set specified in (b) into the context of Program I becomes those $(p_o, p) \in \mathbb{R} \times \mathbb{R}^m$ for which there exist $y \in \mathbb{R}^{(\mathcal{S})}$, $\lambda \in \mathbb{R}^{(\mathcal{U})}$, $\omega \in \mathbb{R}$ satisfying

$$0 = \sum_\gamma u_o(\gamma)\lambda(\gamma) \tag{27}$$

$$A^T p = \sum_\beta b(\beta)y(\beta) + \sum_\gamma u(\gamma)\lambda(\gamma) \tag{28}$$

$$p_o = -\sum_\beta b_{n+1}(\beta)y(\beta) - \sum_\gamma u_{n+1}(\gamma)\lambda(\gamma) + \omega \tag{29}$$

and $y \geq 0$, $\lambda \geq 0$, $\omega \geq 0$.

The task is to show that any (p_o, p) satisfying (25)-(29) must necessarily be $\underset{\sim}{0}$ under the hypotheses of Theorem 1 above.

Observe that $u_o(\gamma) \geq 0$ for all $\gamma \in \mathcal{U}$, and therefore $\lambda(\gamma) = 0$ when $u_o(\gamma) > 0$. Thus (27) is eliminated and some of the variables in (28) and (29). We now take care of those $\lambda(\gamma)$ for which $u_o(\gamma) = 0$.

Let $\mathcal{U}_o = \{\gamma \in \mathcal{U} \mid u_o(\gamma) = 0\}$. As observed above, (28) becomes

$$A^T p = \sum_\beta b(\beta)y(\beta) + \sum_{\gamma \in \mathcal{U}_o} u(\gamma)\lambda(\gamma). \tag{30}$$

By Assumption 2 for each $\gamma \in \mathcal{U}_o$ there exists a positive scalar k_γ and some $\beta \in \mathcal{S}$ such that

$$k_\gamma u(\gamma) = b(\beta).$$

For each $\beta \in \mathcal{S}$, let

$$\mathcal{S}_\beta = \{\gamma \in \mathcal{U}_o \mid \text{there exists } k_\gamma > 0,\ k_\gamma u(\gamma) = b(\beta)\}. \tag{31}$$

Then $\bigcup_\beta \mathcal{S}_\beta = \mathcal{U}_o$, even though some $\mathcal{S}_\beta$ may be empty. By taking "$\sum_{\gamma \in \mathcal{S}_\beta}$" to mean "$\sum_{\{\gamma \mid \lambda(\gamma) \neq 0, \mathcal{S}_\beta \neq \emptyset, \gamma \in \mathcal{S}_\beta\}}$", a finite sum, we rewrite (30) as

$$A^T p = \sum_\beta b(\beta)\,[y(\beta) + \sum_{\gamma \in \mathcal{S}_\beta} \frac{\lambda(\gamma)}{k_\gamma}]. \tag{32}$$

Similarly (29) becomes

$$p_o = -\sum_\beta b_{n+1}(\beta)\,[y(\beta) + \sum_{\gamma \in \mathcal{S}_\beta} \frac{\lambda(\gamma)}{k_\gamma}] + \omega. \tag{33}$$

Together (32) and (33) imply

$$\begin{pmatrix} A^T p \\ p_o \end{pmatrix} \in C_Q, \tag{34}$$

while we are also given that $p \in O^+ P$ and $(gO^+)(p) \geq p_o$. According to the fourth assumption in the theorem it must be the case that $(p_o, p) = \underset{\sim}{0}$. Therefore by Theorem A1, Program I is consistent, $V_I = V_{II}$ and V_I is a maximum. Applying Lemma 2 yields $V_I = V_M = V_N = V_{II}$.

<u>Corollary 1.1.</u> The cone C_Q is closed.

<u>Proof</u>. Suppose $(a,a_{m+1}) \in \mathbb{R}^m \times \mathbb{R}$ is a limit point of the cone C_Q. Then it is a limit point of the entire cone specified in Assumption 3.2, which is closed by assumption. Hence there is an expression of $(\underset{\sim}{Q},a,a_{m+1})$ as the left-hand side of the equations (27), (28), (29) in terms of the right-hand side as indicated. The above algebraic argument then applies to show completely analogous to (34), that $\binom{a}{a_{m+1}} \in C_Q$.

Another corollary of Theorem 1 is the following complementary slackness result, which is merely a translation of Theorem 2 of [6] in the context of Programs I and II.

<u>Corollary 1.2.</u> Let $\{p_o^*,p^*,y^*,\lambda^*\}$ be optimal for Program I and $\{q_o^*,q^*,x^*,\eta^*\}$ be optimal for Program II. Then

$$x^*(\alpha)[d(\alpha)^T p^* - d_{m+1}(\alpha)] = 0 \quad \text{for all} \quad \alpha\in\mathcal{R} \tag{35}$$

$$y^*(\beta)[b(\beta)^T q^* - b_{n+1}(\beta)] = 0 \quad \text{for all} \quad \beta\in\mathcal{S} \tag{36}$$

$$\eta^*(\sigma)[p_o^* v_o(\sigma) + p^{*T} v(\sigma) - v_{m+1}(\sigma)] = 0 \text{ for all } \sigma\in\mathcal{J} \tag{37}$$

$$\lambda^*(\gamma)[u_o(\gamma)q_o^* + u(\gamma)^T q^* - u_{n+1}(\gamma)] = 0 \text{ for all } \gamma\in\mathcal{U}. \tag{38}$$

As Theorem 2A is a companion to and actually a corollary of Theorem 1A, we obtain the natural companion to Theorem 1.

<u>Theorem 2</u>. Let Assumptions 1-3 prevail. Assume that $(q_o,q) \in \mathbb{R} \times \mathbb{R}^n$, $q \in O^+Q$, $(hO^+)(q) \leq q_o$, and

$$\begin{pmatrix} Aq \\ q_o \end{pmatrix} \in C_P \ \underline{\text{implies}} \ (q_o,q) = \underset{\sim}{0}.$$

Then $V_I = V_M = V_N = V_{II}$ and V_{II} is a minimum.

Analogous to Corollary 1.1, the cone C_P is also closed.

<u>Theorem 3</u>. Let Assumptions 1-3 prevail. Assume

(i) $p \in O^+P$, $(gO^+)(p) \geq p_o$ and

$$\begin{pmatrix} A^Tp \\ p_o \end{pmatrix} \in C_Q \ \text{implies} \ (p_o,p) = \underset{\sim}{0}$$

<u>and</u>

(ii) $q \in O^+Q$, $(hO^+)(q) \leq q_o$ and

$$\begin{pmatrix} Aq \\ q_o \end{pmatrix} \in C_P \ \text{implies} \ (q_o,q) = \underset{\sim}{0}.$$

Then I has an optimal solution $\{p^*,g(p^*),y^*,\lambda^*\}$ and II has an optimal solution $\{q^*,h(q^*),x^*,\eta^*\}$ and for any such solution p^*,q^* is a saddle solution, i.e.

$$V_M = V_N = g(p^*) + p^{*T}Aq^* + h(q^*).$$

<u>Proof</u>. By Theorems 1 and 2 optimal solutions exist and by Corollary 1 equations (35)-(38) are satisfied.

Analogous to the proof of Lemma 2 we see that

$$p^{*T}Aq^* = \sum_\beta y^*(\beta)b(\beta)^T q^* + \sum_\gamma \lambda^*(\gamma)u(\gamma)^T q^* . \qquad (39)$$

Applying (36) and (38) to (39) yields

$$p^{*T}Aq^* = \sum_\beta y^*(\beta)b_{n+1}(\beta) - q_o^* + \sum_\gamma u_{n+1}(\gamma)\lambda^*(\gamma).$$

Adding $p_o^* = g(p^*)$ to both sides and transposing $q_o^* = h(q^*)$, we obtain

$$\begin{aligned} g(p^*) + p^{*T}Aq^* + h(q^*) &= \sum_\beta y^*(\beta)b_{n+1}(\beta) + \sum_\gamma u_{n+1}(\gamma)\lambda^*(\gamma) + p_o^* \\ &= V_I = V_{II} = V_M = V_N . \end{aligned}$$

Theorem 3 is the main saddle value theorem of the paper. The conditions underlying Theorem 3 and Theorems 1 and 2 are among the common ones in the literature which guarantee finite saddle-values and existence of saddle-points for the case when the sets P and Q are unbounded. Roughly speaking when conditions prevail to guarantee saddle points, then each of our dual separably-infinite programs will have optimal solutions. We briefly review some of the conditions in the literature.

5. Relation of the Assumptions of Theorem 3 to the Literature

Among rather general conditions sufficient for guaranteeing a saddle-point are (a) and (b) of Theorem 37.3 in Rockafellar [15]. These conditions apply to more general concave-convex saddle functions than the particular one we have studied, namely

$$K(p,q) = g(p) + p^T A q + h(q)$$

for $(p,q) \in P \times Q$.

On the other hand, our theorems go beyond a saddle-point result, namely showing the equivalence of a saddle-value solution to the solution of two dual separably-infinite linear programs.

In the context of our saddle function, condition (a) of Theorem 37.3 is equivalent to our fourth assumption in Theorem 2.

In the case at hand, condition (a) relates to directions of recession of the convex function $K(p,\cdot)$, where $p \in P$. A direction of recession of this function, say $\bar{q}$, satisfies

$$p^T A\bar{q} + (hO^+)(\bar{q}) \leq 0.$$

Condition (a) states that

$$p^T A\bar{q} + (hO^+)(\bar{q}) \leq 0 \quad \text{for all} \quad p \in \text{ri } P \tag{40a}$$

<u>implies</u>

$$p = 0. \tag{40b}$$

Now in (40a), ri P may be replaced with P, and hence by Lemma 1 of the Appendix, (40a) is equivalent to

$$\begin{pmatrix} A\bar{q} \\ (hO^+)(\bar{q}) \end{pmatrix} \in \mathrm{cl}\ C_P.$$

By Corollary 1.1, C_P is closed, and it is now straightforward to check that Assumption 4 of Theorem 2, namely: $q \in O^+Q$, $(hO^+)(q) \leq 0$ and

$$\begin{pmatrix} Aq \\ q_o \end{pmatrix} \in C_P \text{ implies } (q_o, q) = \underset{\sim}{0},$$

is equivalent to condition (a) of Theorem 37.3 [15].

Similarly, condition (b) is equivalent to the fourth assumption of Theorem 1.

In the more general context of infinite linear programming Fan introduced related asymptotic conditions and proved them to be sufficient for duality theorems, [9].

6. A Simple Example of an Equilibrium in Economics

The following example is related to work of Charnes and Cooper, Charnes and Carey, and Charnes and Thore concerning equilibria in resource value-transfer economies [4] and private ownership economies of Debreu type [8].

Let p_1 and p_2 denote respectively the price of a consumer good and wages paid for work done to produce the single good. As an output the consumer good y_1 shall be non-negative, while the single input y_2 shall be non-positive. The set of possible (y_1,y_2) combinations shall be termed the production set Y, and shall be taken to be:

$$Y = \{(y_1,y_2) \in \mathbb{R}^2 \mid y_1 \leq \sqrt{-y_2},\ y_2 \leq 0\}.$$

A system of linear supports for Y is the following:

$$\{(y_1,y_2) \mid y_1 + \frac{1}{2\sqrt{-\alpha}}\, y_2 \leq \frac{1}{2}\sqrt{-\alpha} \quad \text{for all} \quad \alpha < 0\}. \tag{41}$$

The consumer shall be guided by a potential function of p_1, p_2, given by

$$E(p_1,p_2) = \sqrt{p_1} - 1/3\, p_2.$$

For a given list of positive prices $(\bar{p}_1,\bar{p}_2)$ it shall be assumed that the consumer demands

$$\frac{\partial E}{\partial p_1}(\bar{p}_1,\bar{p}_2) = \frac{1}{2}\frac{1}{\sqrt{\bar{p}_1}} \quad \text{of the consumer good,} \tag{42}$$

while he is willing to supply

$$\left|\frac{\partial E}{\partial p_2}(\bar{p}_1,\bar{p}_2)\right| = \left|-\frac{1}{3}\right| = 1/3 \qquad (43)$$

units of labor. (Supply is taken to be a negative number.)

Taking $h(p_1,p_2) = -E(p_1,p_2)$, we may consider the following support system[1] for the epigraph of the convex function h: (q_o,p_1,p_2),

$$q_o + p_1 \frac{1}{2}\frac{1}{\sqrt{\gamma}} - \frac{1}{3}p_2 \geq -\frac{1}{2}\sqrt{\gamma}\ , \ \underline{\text{for all}}\ \ \gamma > 0. \qquad (44)$$

Viewing the producer as a profit maximizer we consider the following polyextremal problem for the economy.

<u>Find</u> $\qquad V_M = \sup_{y}\ \inf_{p}\ p_1y_1 + p_2y_2 - \sqrt{p_1} + 1/3\ p_2$

<u>from among</u> $\qquad y \in \mathbb{R}^2,\ p \in \mathbb{R}^2$ <u>satisfying</u>

$$y_1 \leq \sqrt{-y_2}\ ,$$

$y_2 \leq 0$ <u>and</u> $p_1 \geq 0,\ p_2 \geq 0.$

Let us now give the dual separably infinite programs appropriately identified with the producer and consumer respectively.

<u>Producer's Separably Infinite Program</u> (I)

<u>Find</u> $V_I = \sup_{\alpha} \Sigma - \frac{1}{2}\sqrt{\gamma}\ \lambda(\gamma)$ <u>from among</u> $y \in \mathbb{R}^2$ <u>and</u> λ (45)
<u>a generalized finite sequence on</u> $\mathbb{R}_{(>0)}$ <u>which satisfy</u>

[1]At this point we are not insuring that the relevant moment cones are closed as for example if (41) and (44) were canonically closed, see [10], p. 12, and [15], page 200.

$$y_1 + \frac{1}{2\sqrt{-\alpha}}\, y_2 \leq \frac{1}{2}\sqrt{-\alpha}, \quad \text{all} \quad \alpha < 0 \tag{46}$$

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \sum_{\gamma} \begin{pmatrix} \frac{1}{2}\frac{1}{\sqrt{\gamma}} \\ -1/3 \end{pmatrix} \lambda(\gamma) - \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{47}$$

$$\sum_{\gamma} \lambda(\gamma) = 1 \tag{48}$$

<u>and</u>

$$\xi_1, \xi_2 \geq 0, \; y_2 \leq 0, \; \lambda \geq 0. \tag{49}$$

An optimal solution to (I) is:

$$\gamma^* = 3/4, \quad \lambda^*(\gamma^*) = 1, \quad \lambda^*(\gamma) = 0 \quad \text{for} \quad \gamma \neq \gamma^*, \quad \xi_1^* = \xi_2^* = 0,$$

and

$$y_1^* = \frac{1}{\sqrt{3}}, \quad y_2^* = -\frac{1}{3} \quad \text{with} \quad V_I = -\frac{\sqrt{3}}{4}.$$

Observe that the producer does not need to know the specific prices which the consumer will select.

<u>Consumer's Separably Infinite Program</u> (II)

<u>Find</u> $V_{II} = \inf q_o + \sum_{\alpha<0} \frac{1}{2}\sqrt{-\alpha}\, x(\alpha)$ <u>from among</u> $p \in \mathbb{R}^2$ <u>and</u> (50)

x <u>a generalized finite sequence on</u> $\mathbb{R}_{(<0)}$, <u>which satisfy</u>

$$q_o + p_1 \frac{1}{2\sqrt{\gamma}} - \frac{1}{3} p_2 \geq -\frac{\sqrt{\gamma}}{2}, \quad \text{for all} \quad \gamma > 0 \tag{51}$$

and

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} - \sum_{\alpha<0} \begin{pmatrix} 1 \\ \frac{1}{2\sqrt{-\alpha}} \end{pmatrix} x(\alpha) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{52}$$

and

$$p_1, p_2 \geq 0, \quad x \geq 0. \tag{53}$$

An optimal solution to (II) is:

$$\alpha^* = -\tfrac{1}{3}, \quad x^*(\alpha^*) = 3/4, \quad x^*(\alpha) = 0 \quad \text{for} \quad \alpha \neq \alpha^*$$

and

$$p_1^* = 3/4, \quad p_2^* = \frac{3\sqrt{3}}{8}, \text{ with } V_{II} = \frac{-\sqrt{3}}{4}.$$

Note that the consumer's problem determines the full set of prices p_1^*, p_2^*.

Producer consumer equilibria is checked as follows.

Producer Equilibrium. Taking prices $p_1^* = 3/4$ and $p_2^* = \frac{3\sqrt{3}}{8}$, the producer seeks

$$\begin{array}{ll} \max & p_1^* y_1 + p_2^* y_2 \\ \text{subject to} & y_1 \leq \sqrt{-y_2}, \quad y_2 \leq 0. \end{array}$$

This problem has a unique solution $\hat{y}_1 = \frac{1}{\sqrt{3}}$ and $\hat{y}_2 = -\hat{y}_1^2 = -\frac{1}{3}$. Since $\hat{y}_1 = y_1^*$ and $\hat{y}_2 = y_2^*$, the producer is in equilibrium.

Consumer Equilibrium

According to (42) and (43), at prices p_1^*, p_2^* the consumer demands

$$\frac{\partial E}{\partial p_1}(p_1^*, p_2^*) = \frac{1}{2\sqrt{p_1^*}} = \frac{1}{\sqrt{3}}$$

and is willing to supply (a negative number by convention)

$$\frac{\partial E}{\partial p_2}(p_1^*, p_2^*) = -\frac{1}{3}.$$

Since $\frac{\partial E}{\partial p_1}(p_1^*, p_2^*) = y_1^*$ and $\frac{\partial E}{\partial p_2}(p_1^*, p_2^*) = y_2^*$, the consumer is also in equilibrium.

An important conclusion from this simple example is that each of the problems I and II has its own individual character. The basic purpose of the producer's problem I is to determine outputs and inputs. It may derive implicitly some or all of the optimal prices of outputs and inputs. Determining the full set of prices is however, the main thrust of the consumer's problem II.

Moreover, the parametric form of each player's decision vector in terms of generalized finite sequences has an economic interpretation. For the producer, (47) says that an output-input vector y must be a convex combination of the consumer's vectors of demand and supply, $\nabla E(\gamma)$. For the consumer, (52) says that the price vector p should be a convex combination of vectors normal to the frontier of the production set.

7. Conclusions

In this paper a new approach has been developed to solve polyextremal problems, specifically the "maximin" and "minimax" variety. Conceptually, the procedure is to decouple or decentralize the jointly defined problem into two dual separably infinite programs. These infinite programs possess special structure amenable to numerical treatment by semi-infinite programming methods, and do not involve any internal extremizations.

Basically the structure requires an augmentation of each player's original variables by generalized finite sequences determined by the other player's information sets. The conditions under which the polyextremal problem may be solved by two dual linear programs are among the most general appearing in the literature, permitting for example, unboundedness in each of the explicit constraint sets.

The procedure has been illustrated in the solution of a simple producer-consumer polyextremal economy, where the producer seeks to maximize profits and the consumer seeks to minimize expenditures while achieving some implicit satisfaction level. Each of the two players has his own separably infinite program. The producer's program determines optimal inputs and outputs, while the consumer's program determines optimal prices of inputs and outputs without specific knowledge of the producer's optimal decisions. Thus, each player modifies his choices in the light of constraints and goals associated with the actions of the other player, but he does not require knowledge of the other players choice itself.

The example was motivated by work of Charnes and Carey, Charnes and Cooper, and Charnes and Thore on problems of economic equilibrium. Further work is planned in this area, in particular, interpreting the assumptions underlying the saddle value theorems in this paper in the context of private ownership and resource value transfer economies. Additional applications are envisioned in physics and engineering.

Appendix: An Extension of a Duality Result of [6]

The duality theorems of this appendix could be derived as corollaries of infinite linear programming results of Fan [9]. We present a simple derivation based on elementary separation in finite dimensions. Consider the following dual pair of separably infinite programs.

Program P. Let $S \subseteq R^k$, $Q \subseteq R^{\ell}$ and let $u(\cdot) : S \to R^n$, $u_{n+1}(\cdot) : S \to R$, $v(\cdot) : Q \to R^m$, $v_{m+1}(\cdot) : Q \to R$, $c \in R^n$, $b \in R^m$ and $A \in R^{m \times n}$.

Find $V_P = \sup \sum_{t \in S} u_{n+1}(t)\lambda(t) - b^T y$

from among $\lambda(\cdot) \in R^{(S)}$ and $y \in R^m$ which satisfy

$$\sum_{t \in S} u(t)\lambda(t) - A^T y = c \tag{1a}$$

$$v^T(r)y \leq v_{m+1}(r) \text{ for all } r \in Q \tag{1b}$$

and

$$\lambda(\cdot) \geq 0 \tag{1c}$$

and

Program D

Find $V_D = \inf c^T x + \sum_{r \in Q} v_{m+1}(r)\eta(r)$

from among $x \in R^n$ and $\eta(\cdot) \in R^{(Q)}$ which satisfy

$$u^T(t)x \geq u_{n+1}(t) \text{ for all } t \in S \tag{2a}$$

$$-Ax + \sum_{r \in Q} v(r)\eta(r) = -b \tag{2b}$$

and

$$\eta(\cdot) \geq 0. \tag{2c}$$

The following convex sets will be used in the duality developments.

$$K_Q \equiv \{y \in R^m \mid v^T(r)y \le v_{m+1}(r) \text{ for all } r \in Q\},$$

$$K_S \equiv \{x \in R^n \mid u^T(t)x \ge u_{n+1}(t) \text{ for all } t \in S\},$$

$C_Q \subseteq R^{m+1}$ is the convex cone generated by

$$\left\{\begin{pmatrix} v(r) \\ -v_{m+1}(r) \end{pmatrix} \middle| \, r \in Q\right\} \cup \left\{\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\}.$$

$C_S \subseteq R^{n+1}$ is the convex cone generated by

$$\left\{\begin{pmatrix} u(t) \\ -u_{n+1}(t) \end{pmatrix} \middle| \, t \in S\right\} \cup \left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\},$$

C_S and C_Q are often called <u>moment</u> <u>cones</u>.

The following lemma is known but its proof is included for completeness.

<u>Lemma</u> <u>1</u>. Assume that K_Q and K_S are non-empty.

Then

(1a) $\binom{g}{-h} \in \text{cl } C_Q$ (closure of C_Q) if and only if $\omega^T g \le h$ for all $\omega \in K_Q$

and

(1b) $\binom{g}{-h} \in \text{cl } C_S$ if and only if $\omega^T g \ge h$ for all $\omega \in K_S$.

<u>Proof</u>. The proof of both parts is similar, so (1b) is proven. We first show for fixed $g \in \mathbb{R}^n$, $h \in \mathbb{R}$ that the following statement

$$"\omega \in \mathbb{R}^n,\ \omega_{n+1} \in \mathbb{R},\ \omega^T g - \omega_{n+1} h \geq 0 \tag{3a}$$

whenever

$$\omega^T u(t) - \omega_{n+1} u_{n+1}(t) \geq 0, \quad \text{for each} \quad t \in S \tag{3b}$$

$$\text{and } \omega_{n+1} \geq 0" \tag{3c}$$

<u>is equivalent to</u>

$$"\omega \in \mathbb{R}^n,\ \omega^T g \geq h \tag{4a}$$

whenever

$$\omega^T u(t) \geq u_{n+1}(t), \quad \text{for each} \quad t \in S". \tag{4b}$$

To prove the equivalence assume the first statement [(3a)-(3c)] and let $\bar{\omega} \in \mathbb{R}^n$ satisfy (4b). Merely setting $\bar{\omega}_{n+1} = 1$ yields that $(\bar{\omega}, \bar{\omega}_{n+1})$ satisfies (3b) and (3c) and hence by (3a)

$$\bar{\omega}^T g - h \geq 0 \text{ proving (4a).}$$

Let us now assume the second statement to be valid. Let $\bar{\omega} \in \mathbb{R}^n$, $\bar{\omega}_{n+1} \in \mathbb{R}$ satisfy (3b) and (3c). We wish to show (3a) holds. Two cases arise.

<u>Case 1</u> $\bar{\omega}_{n+1} > 0$. In this case $\omega = \bar{\omega}/\bar{\omega}_{n+1}$ satisfies (4b) and hence by assumption must satisfy (4a) which in turn implies $\bar{\omega}^T g - \bar{\omega}_{n+1} h \geq 0$, i.e., (3a) holds in this case.

<u>Case 2</u> $\bar{w}_{n+1} = 0$. In this case $\bar{w}^T u(t) \geq 0$ for each $t \in S$. Since K_S is non-empty there exists $\hat{w} \in K_S$, and so it follows that $\hat{w} + M\bar{w}$ satisfies (4b) for M arbitrarily large. Therefore, by assumption,

$$(\hat{w} + M\bar{w})^T g \geq h, \quad \text{for any positive } M. \tag{5}$$

But it follows from (5) that $\bar{w}^T g \geq 0$, which is (3a) for this particular case. Therefore in this case also we have shown (3a) holds whenever (3b) and (3c) do. Hence the first statement follows from the second, and they are therefore equivalent.

An infinite version of the Farkas Lemma states that

$$\binom{g}{-h} \in \text{cl } C_S$$

if and only if (3a) holds whenever (3b) and (3c) hold, see [15]. The equivalence established above provides the equivalence required in the statement of the lemma.

For a given non-empty convex set K, the <u>recession cone of</u> K, denoted 0^+K is the set of all vectors y such that $x + \lambda y \in K$ for every $\lambda \geq 0$ and $x \in K$, see [15]. For a linear transformation A, the subspace of all vectors z such that $Az = 0$, is termed the <u>kernel</u> of A and denoted $\ker A$.

With these preliminary definitions we are ready to prove the first of two symmetric theorems.

<u>Theorem A1</u>. Assume that Program D is consistent and finite valued and that the convex cone C_S is closed. Let the following assumption prevail

$$\text{(A1)} \qquad 0^+K_Q \cap \{y \in \mathbb{R}^m \mid \begin{pmatrix} A^T y \\ -b^T y \end{pmatrix} \in C_S\} = \{0\}.$$

Then Program P is consistent, $V_P = V_D$, and V_P is a maximum.

__Proof__. Program D can be written in the following form.

__Find__ $V_D = \inf z$

__from among__ $x \in R^n$, $\eta(\cdot) \in R^{(Q)}$, $z \in R$, $w \in R$ __which satisfy__

$$u^T(t)x \geq u_{n+1}(t) \ \underline{\text{for all}} \quad t \in S$$

$$\begin{pmatrix} Ax \\ c^T x \end{pmatrix} + \sum_{r \in Q} \begin{pmatrix} v(r) \\ -v_{m+1}(r) \end{pmatrix} \left[-\eta(r)\right] + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \left[-w\right] = \begin{pmatrix} b \\ z \end{pmatrix}$$

__and__

$$\eta(\cdot) \geq 0, \ w \geq 0.$$

This is equivalent to the following form.

__Find__ $V_D = \inf z$

__from among__ $x \in R^n$, $z \in R$ __which satisfy__

$$x \in K_S$$

$$\begin{pmatrix} b - Ax \\ z - c^T x \end{pmatrix} \in -C_Q.$$

Let us define the set $\overline{K} \subseteq R^{m+1}$

$$\overline{K} = \left\{ \begin{pmatrix} b - Ax \\ z - c^T x \end{pmatrix} \middle| \ x \in K_S \ \text{ and } \ z < V_D \right\}.$$

Since K_S is convex, $\overline{K}$ is also convex. Now V_D is the value of Program D. Thus there cannot be an $\bar{x} \in K_S$ and $\bar{z} < V_D$ such that

$$\begin{pmatrix} b - A\bar{x} \\ \bar{z} - c^T \bar{x} \end{pmatrix} \in -C_Q.$$

Hence

$$\overline{K} \cap \left(-C_Q\right) = \emptyset.$$

Since $\overline{K}$ and $-C_Q$ are disjoint, non-empty convex sets, we can find a hyperplane that separates them ([16] Theorem 3.3.9 or [15] Theorem 11.3). That is, there exist $y \in R^m$ and $y_{m+1} \in R$, not both zero, such that

$$y^T d + y_{m+1} d_{m+1} \geq 0 \quad \underline{\text{for all}} \quad \begin{pmatrix} d \\ d_{m+1} \end{pmatrix} \in -C_Q \tag{6}$$

and

$$y^T(b - Ax) + y_{m+1}(z - c^T x) \leq 0$$

$$\underline{\text{for all}} \quad x \in K_S \quad \underline{\text{and}} \quad z < V_D. \tag{7}$$

Now $\begin{pmatrix} -v(r) \\ v_{m+1}(r) \end{pmatrix} \in -C_Q$ for all $r \in Q$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in -C_Q$. Thus (6) implies that

$$-y^T v(r) + y_{m+1} v_{m+1}(r) \geq 0 \quad \text{for all} \quad r \in Q \tag{8}$$

$$y_{m+1} \geq 0.$$

We consider two cases on the value of y_{m+1}.

Case 1 $y_{m+1} > 0$. We may take $y_{m+1} = 1$. Then (8) and (7) yield:

$$y^T v(r) \leq v_{m+1}(r) \quad \text{for all} \quad r \in Q \tag{9}$$

and

$$y^T(b - Ax) + (z - c^T x) \leq 0 \quad \text{for all} \quad x \in K_S \quad \text{and} \quad z < V_D.$$

This last inequality can be written as

$$x^T(A^Ty + c) \geq b^Ty + z \quad \text{for all} \quad x \in K_S \quad \text{and} \quad z < V_D.$$

By Lemma 1(1b), this is equivalent to

$$\begin{pmatrix} A^Ty + c \\ -b^Ty - z \end{pmatrix} \in C_S \quad \text{for all} \quad z < V_D.$$

Since C_S is closed,

$$\begin{pmatrix} A^Ty + c \\ -b^Ty - V_D \end{pmatrix} \in C_S.$$

Thus there exist a $\lambda(\cdot) \in R^{(S)}$ and $w \in R$ which satisfy

$$A^Ty + c = \sum_{t \in S} u(t)\lambda(t) \tag{10}$$

$$-b^Ty - V_D = \sum_{t \in S} - u_{n+1}(t)\lambda(t) + w \tag{11}$$

$$\lambda(\cdot) \geq 0 \quad \text{and} \quad w \geq 0. \tag{12}$$

From (9), (10), (12) it follows that y, λ is feasible for Program P, and hence $\sum_t u_{n+1}(t)\lambda(t) - b^Ty \leq V_P$. But the duality inequality $[V_P \leq V_D]$ and $V_D \leq \sum_t u_{n+1}(t)\lambda(t) - b^Ty$ from (11) and (12) combine to show that V_P is indeed a finite maximum equal to V_D.

Case 2 $y_{m+1} = 0$. It follows from (8) that

$$y^T v(r) \leq 0 \quad \text{for all} \quad r \in Q. \tag{13}$$

According to the definition of K_Q, (13) means that $y \in 0^+ K_Q$. On the other hand in this case, (7) becomes

$$y^T(b - Ax) \leq 0 \quad \text{for each} \quad x \in K_S. \tag{14}$$

Applying Lemma 1, (1b), to (14) implies

$$\begin{pmatrix} A^T y \\ -b^T y \end{pmatrix} \in C_S,$$

since C_S is closed by assumption. Hence

$$y \in 0^+ K_Q \cap \left\{ y \,\middle|\, \begin{pmatrix} A^T y \\ -b^T y \end{pmatrix} \in C_S \right\}$$

and therefore by assumption (A1), $y = 0$. Hence Case 2 cannot happen because $(y, y_{m+1}) \neq 0$. Therefore only Case 1 can occur, completing the proof of Theorem 1.

Theorem A1 has a companion starting with consistency of Program P. It can be proved by rewriting P as a minimization under appropriate variable changes and applying Theorem 1.

Theorem A2. Assume that Program P is consistent and finite valued and that the convex cone C_Q is closed. Let the following property prevail.

(A2) $$0^+ K_S \cap \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{pmatrix} Ax \\ c^T x \end{pmatrix} \in C_Q \right\} = \{0\}.$$

Then Program D is consistent, $V_P = V_D$, and V_D is a minimum.

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13. ABSTRACT

As a direct extension of Charnes' characterization of two-person zero-sum constrained games by linear programming, we show how a general class of saddle value problems can be reduced to a pair of uniextremal dual separably-infinite programs. These programs have an infinite number of variables and an infinite number of constraints, but only a finite number of variables appear in an infinite number of constraints and only a finite number of constraints have an infinite number of variables. The conditions under which the characterization holds are among the more general ones appearing in the literature sufficient to guarantee the existence of a saddle point of a concave-convex function.

The key construction involves augmenting a given player's original set of variables by generalized finite sequences determined by the other player's constraint set and objective function. A duality theory is developed which includes complementarity conditions, thereby making contact with the numerical treatment of semi-infinite programming.

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